

CENTRAL EXTENSIONS OF PREPROJECTIVE ALGEBRAS, THE QUANTUM HEISENBERG ALGEBRA, AND 2-DIMENSIONAL COMPLEX REFLECTION GROUPS

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1. INTRODUCTION

Preprojective algebras of quivers were introduced in 1979 by Gelfand and Ponomarev [GP], because for quivers of finite ADE type, they are models for indecomposable representations (they contain each indecomposable exactly once). Twenty years later, these algebras and their deformed versions introduced in [CBH] (for arbitrary quivers) became a subject of intense interest, since their representation varieties, called quiver varieties, played an important role in geometric representation theory. Ironically, it is exactly for quivers of finite ADE type that preprojective algebras fail to have good properties – they are not Koszul and their deformed versions are not flat.

One of the goals of this paper is to partially correct this problem. We do so by introducing a central extension of the preprojective algebra of a finite Dynkin quiver (depending on a regular weight for the corresponding root system), whose natural deformed version is actually flat, although it ceases to be flat after factorization by the central element.¹ We calculate the Hilbert polynomial of the central extension, and show that it is a Frobenius algebra. As a corollary, we obtain the Hilbert series of the usual deformed preprojective algebra in which the deformation parameters are variables, and show that this algebra is Gorenstein (although it is not a flat module over the ring of parameters).

The main tool in the proofs is the fact that our central extension for the weight ρ is the image of the quantum Heisenberg algebra in the fusion category of representations of quantum SL_2 under a tensor functor into R -bimodules (where R is the algebra of idempotents of the quiver). This is a generalization of the result of [MOV] which says that the usual preprojective algebra is the image of the quantum symmetric algebra under the same functor.

We also construct Riemann-Hilbert homomorphisms from the cyclotomic Hecke algebras of certain 2-dimensional complex reflection groups to the “spherical” subalgebras of the deformed central extensions of preprojective algebras. This allows us to show that if all parameters of the cyclotomic Hecke algebra are equal to 1 (i.e. the generators are unipotent) then the

¹We note that our construction makes sense for any quiver, but produces something new only in the Dynkin case.

block of the trivial representation is equivalent to the category of representations of the spherical subalgebra of the centrally extended preprojective algebra. As a by-product, we show that the dimension of the cyclotomic Hecke algebra of a 2-dimensional complex reflection group for generic parameters is equal to the order of the group, as conjectured by Broue, Malle, and Rouquier.

The organization of the paper is as follows. In Section 2 we recall basic facts about deformed preprojective algebras, Frobenius algebras, and Cohen-Macaulay and Gorenstein algebras. In Section 3 we state the main results regarding central extensions of preprojective algebras and their deformations, as well as the corresponding spherical subalgebras. In Section 4 we set up the machinery of quantum SL_2 and quantum Heisenberg algebra, which we use in Section 5 to prove the results of Section 2. Finally, in Section 6 we introduce and study the Riemann-Hilbert homomorphism.

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2. PRELIMINARIES

2.1. Quivers and deformed preprojective algebras. Let Q be a quiver of finite Dynkin (that is, ADE) type with Cartan matrix A , and Coxeter number h . Let $I = \{1, \dots, r\}$ be the vertex set of Q , and $R = \bigoplus_{i \in I} \mathbb{C}e_i$ be the commutative algebra generated by idempotents e_i with $e_i e_j = 0$, $i \neq j$. Thus any element $\mu \in R$ can be written as $\mu = \sum_{i \in I} \mu_i e_i$, $\mu_i \in \mathbb{C}$.

Let $x_i \in R^*$ be the dual basis to e_i . It is well known that x_i span a root system Δ inside R^* with Cartan matrix A . We will identify R and R^* using the Weyl group invariant inner product such that $(x_i, x_i) = 2$. Then e_i are the fundamental weights of Δ .

Let $\rho \in R$ be the sum of fundamental weights. Of course, $\rho = 1$, but we'll use the notation ρ to emphasize the connection with Lie theory.

Let $\overline{Q} = Q \cup Q^*$ be the double of Q , – the quiver with the same vertex set and an additional opposite edge $a^* \in Q^*$ for every edge $a \in Q$.

The generic deformed preprojective algebra Π of Q ([CBH],[Ru]) is the quotient of the path algebra of \overline{Q} over the ring $\mathbb{C}[x_1, \dots, x_r] = S(R^*)$ by the relation

$$\sum_{a \in Q} [a, a^*] = \sum_{i \in I} x_i e_i$$

(here $\sum x_i e_i$ is the canonical element of $R^* \otimes R$). This algebra is graded ($\deg(e_i) = 0$, $\deg(x_i) = 2$, $\deg(a) = \deg(a^*) = 1$). Also, it is clear that Π is independent on the orientation of the graph Q , up to an isomorphism: the reversal of orientation of an edge a may be accomplished by replacing a by $-a$.

Let Π_0 be the zero fiber of Π , i.e. the quotient of the path algebra of $\mathbb{C}\overline{Q}$ of \overline{Q} by the relation $\sum_{a \in Q} [a, a^*] = 0$. It is called the (Gelfand-Ponomarev) preprojective algebra of Q . The fiber Π_λ of Π at $\lambda \in R$ (i.e. the quotient of the path algebra of $\mathbb{C}\overline{Q}$ of \overline{Q} by the relation $\sum_{a \in Q} [a, a^*] = \lambda$) is called the deformed preprojective algebra.

It is known that Π_0 is finite dimensional. Namely, as pointed out already by Gelfand and Ponomarev, Π_0 is a model for indecomposable representations of Q (i.e., the direct sum of all of them taken once). Thus the dimension of Π_0 is the sum of heights of all positive roots:

$$\dim(\Pi_0) = \sum_{\alpha > 0} (\alpha, \rho) = 2(\rho, \rho) = \frac{h(h+1)r}{6}.$$

(see [MOV]). The last equality follows from Freudenthal's magic formula $((\rho, \rho) = h \dim \mathfrak{g}/12$, where \mathfrak{g} is the simple Lie algebra attached to A) and Kostant's formula $((h+1)r = \dim \mathfrak{g})$. Moreover, it is known (see [MOV]) that the Hilbert polynomial of Π_0 with respect to its grading is

$$(1) \quad H_0(t) = \frac{1 + Pt^h}{1 - Ct + t^2},$$

where P is the permutation of fundamental weights corresponding to taking the dual representation of \mathfrak{g} , and C is the adjacency matrix of \overline{Q} .

This implies that Π_λ is finite dimensional for each λ , and Π is a finitely generated $\mathbb{C}[x_1, \dots, x_r]$ -module.

It is known [CB1] that unlike the non-Dynkin case, the algebra Π is not a free $\mathbb{C}[x_1, \dots, x_r]$ -module. More specifically, it is known (see e.g. [CB1], Theorem 1.2) that the algebra Π_λ is zero unless λ belongs to a reflection hyperplane.

2.2. Frobenius algebras. We recall the basic facts about Frobenius algebras.

Let A be a finite dimensional (unital) algebra over \mathbb{C} . Recall that it is called a Frobenius algebra if $A \simeq A^*$ as a left A -module. This is equivalent to saying that there exists a linear function $f : A \rightarrow \mathbb{C}$ such that the bilinear form $(a, b) = f(ab)$ is nondegenerate. Indeed, given A with an isomorphism of left modules $\phi : A \rightarrow A^*$, we set $f = \phi(1)$, and conversely, given f , we define ϕ by $\phi(a)(b) = f(ba)$.

Lemma 2.1. *Let A be a \mathbb{Z}_+ -graded finite dimensional algebra, $A = \bigoplus_{j \geq 0} A[j]$ such that $R := A[0]$ is a commutative semisimple algebra, and the Hilbert polynomial $P_A(t)$ satisfies the condition $P_A(t) = t^d P_A(t^{-1})$. Then the following conditions are equivalent:*

- (i) *A is Frobenius;*
- (ii) *$A[d]$ is an invertible $A[0]$ -bimodule, and the multiplication map $A[i] \otimes_R A[d-i] \rightarrow A[d]$ defines an isomorphism $A[d-i] \rightarrow A[i]^* \otimes_R A[d]$.*

Proof. Let $R = \bigoplus_{j=1}^r R_j$, where R_j are copies of \mathbb{C} . Then any R -bimodule M can be written as $M = \bigoplus M_{ij} \otimes R_{ij}$, where $R_{ij} = \mathbb{C}$ is the unique irreducible (R_i, R_j) -bimodule.

Suppose (ii) holds. Since $A[d]$ is invertible, we have $A[d] = \bigoplus_q R_{\sigma(q)q}$ for some permutation σ . Then the multiplication map defines an isomorphism

$$A[d - i]_{pq} \rightarrow (A[i]^*)_{p\sigma(q)}.$$

This implies that if $f \in A^*$ vanishes on degrees $< d$, and $f : A[d] \rightarrow \mathbb{C}$ is given by $f(y_1, \dots, y_r) = \sum_{j=1}^r y_j$ ($y_p \in R_{p\sigma(p)}$), then $f(ab)$ is a nondegenerate form on A . Thus (i) holds.

Now suppose (i) holds. Then the map $(a, b) \mapsto f(ab)$ defines pairings $A[0] \otimes A[d] \rightarrow \mathbb{C}$ and $A[d] \otimes A[0] \rightarrow \mathbb{C}$ which have trivial kernel in $A[d]$. Since the dimensions of $A[0]$ and $A[d]$ are the same, these pairings are nondegenerate. This implies that $A[d]$ is faithful as a left and right R -module, which by dimension counting implies that $A[d]$ is invertible, i.e. $A[d] = \bigoplus R_{\sigma(q)q}$. Now the multiplication map defines a linear map $\psi_{ipq} : A[d - i]_{pq} \rightarrow (A[i]^*)_{p\sigma(q)}$.

Assume that (ii) fails. Then by dimension count for some i, p, q the map ψ_{ipq} has a nontrivial kernel. Let us choose the largest such i (clearly $i < d$), and let a be an element in the kernel of ψ_{ipq} . It is clear that $A[m]a = 0$ for all $m > 0$, because for any $b \in A[m]_{sp}$, $b \in \text{Ker} \psi_{i+m, sq}$. Let us pick $a' \in A[d]$ such that $f(ca) = f(ca')$ for any $c \in A[0]$. Then $f(c(a - a')) = 0$ for all $c \in A$. Contradiction. Thus (ii) must hold, as desired. \square

2.3. Cohen-Macaulay and Gorenstein algebras. Let us now recall the basic properties of noncommutative Cohen-Macaulay and Gorenstein algebras. (see also [YZ] or the discussion in [EG], beginning of Section 3).

Let A be a \mathbb{Z}_+ -graded algebra over \mathbb{C} , such $R = A[0]$ is finite dimensional and semisimple. We will assume that A is a finitely generated module over a finitely generated graded central subalgebra B , such that $B[0] = \mathbb{C}$. In this case, by Noether's normalization lemma, we may assume that B is a polynomial algebra; we will do so from now on.

In this situation, the dualizing complex of A may be defined by the formula $\mathbb{D}_A = \text{RHom}_B(A, B)$. It can be shown that it does not depend on the choice of B .

Definition 2.2. The algebra A is said to be Cohen-Macaulay if the cohomology of its dualizing complex is concentrated in degree zero, and Gorenstein if this cohomology is furthermore isomorphic to A as a left module.²

Thus we have the following proposition.

Proposition 2.3. (i) A is Cohen-Macaulay if and only if A is a free B -module.

²We note that in the literature there are several nonequivalent definitions of Gorenstein property for noncommutative algebras, which are all equivalent in the commutative case. The definition we are using implies that A has finite injective dimension, but is not equivalent to this requirement. We also note that our definition is not Morita invariant.

(ii) A is Gorenstein if and only if furthermore there exists a B -linear map $f : A \rightarrow B$ such that the map $(a, b) \mapsto f(ab)$ is a nondegenerate pairing $A \times A \rightarrow B$, i.e. defines an isomorphism $A \rightarrow \text{Hom}_B(A, B)$.

This immediately implies the following.

Proposition 2.4. (i) Let A be a finite dimensional algebra. Then A is Gorenstein if and only if it is Frobenius.

(ii) A is Gorenstein if and only if its zero-fiber A_0 (as a B -module) is Gorenstein.³

3. CENTRAL EXTENSIONS OF PREPROJECTIVE ALGEBRAS

3.1. Gorenstein properties and Hilbert series of generic deformed preprojective algebras. One of the main results of this paper is the following theorem.

Theorem 3.1. (i) Π is a Gorenstein algebra (in particular, Cohen-Macaulay).

(ii) The matrix Hilbert series of Π (i.e. the matrix consisting of the Hilbert series of $e_i \Pi e_j$) is equal to

$$H(t) = \frac{1 - t^{2h}}{(1 - t^2)^r (1 - Ct + t^2)},$$

where $C = 2 - A$ is the adjacency matrix of \overline{Q} .

Theorem 3.1 is proved in Section 5.

3.2. Central extensions of preprojective algebras. Another result of this paper is a construction of a central extension of the preprojective algebra, whose deformed version is flat, unlike that of the usual preprojective algebra. Namely, let $\Pi[z]$ be the algebra of polynomials of a central variable z with coefficients in Π ; it is graded with $\deg(z) = 2$. Let $\mu \in R$ be a regular weight (i.e., does not belong to any reflection hyperplane $(\alpha, \mu) = 0$). For any weight $\lambda \in R$, let Π_λ^μ be the quotient of $\Pi[z]$ by the relations $x_i = \mu_i z + \lambda_i$. Thus Π_λ^μ is the quotient of the path algebra $\mathbb{C}[z]\overline{Q}$ by the relation

$$\sum_a [a, a^*] = \sum_i (\mu_i z + \lambda_i) e_i.$$

This algebra carries a natural filtration induced by the grading in the path algebra.

Theorem 3.2. (i) $\Pi[z]$ is a finitely generated free module over the algebra $K_\mu := \mathbb{C}[x_1 - \mu_1 z, \dots, x_r - \mu_r z]$.

(ii) The associated graded algebra of Π_λ^μ under the natural filtration is Π_0^μ .

(iii) The algebra Π_0^μ is Gorenstein (=Frobenius). One has $z^{h-1} = 0$, and for generic μ the socle of Π_0^μ is $z^{h-2}R$.

³Here by definition $A_0 = A/JA$, where J is the kernel of the augmentation in B .

(iv) The Hilbert polynomial of Π_0^μ is

$$\tilde{H}(t) = H(t)(1 - t^2)^{r-1} = \frac{1 + t^2 + \dots + t^{2(h-1)}}{1 - Ct + t^2}.$$

(v) The dimension of Π_λ^μ is $h^2(h+1)r/12$, and for generic λ it is semisimple.

Theorem 3.2 is proved in Section 5.

Part (v) of Theorem 3.2 implies that for generic λ the element z is semisimple, and hence

$$\Pi_\lambda^\mu = \oplus_{\alpha > 0} \Pi_{\lambda - \frac{(\lambda, \alpha)}{(\mu, \alpha)} \mu}.$$

Thus Π_λ is semisimple for a generic λ on a reflection hyperplane $(\alpha, \lambda) = 0$ (where α is a positive root). Moreover, by [CB1], Theorem 1.2, it is actually simple: $\Pi_\lambda = \text{Mat}_{(\alpha, \rho)}(\mathbb{C})$. This means that the rational Weyl denominator $\delta(x) := \prod_{\alpha > 0} (\alpha, x) \in \mathbb{C}[x_1, \dots, x_r]$ is zero in Π , so Π is scheme-theoretically supported on the reflection hyperplanes. This was conjectured by Rump [Ru] and proved by Crawley-Boevey in [CB2] using a different method.

3.3. Subalgebras corresponding to nodal vertices. It is interesting to consider “spherical subalgebras” of the above algebras corresponding to nodal vertices.

Namely, let Q be of type A_{2n-1} , D or E , and $p \in I$ be the nodal vertex (= the branching vertex for D and E , and the middle vertex for A_{2n-1}). After removal of p , the quiver becomes a union of “legs”, i.e. Dynkin diagrams of type A_{d_k-1} , $k = 1, \dots, m$ ($m = 2$ or 3). Recall that to such Q one can attach a finite subgroup G of $SO(3)$ generated by elements T_k , $k = 1, \dots, m$, with defining relations $T_k^{d_k} = 1$ for $k = 1, \dots, m$, and $\prod_{k=1}^m T_k = 1$ (for type A we get the cyclic group, for type D the dihedral group, for type E_6 the tetrahedral group, for type E_7 the cube group, and for type E_8 the icosahedral group); all finite subgroups of $SO(3)$ are obtained in this way.

The order $|G|$ of the group G can be represented as a product of two integers $q_1 \leq q_2$ such that $q_1 + q_2 - 1 = h/2$. Namely (see e.g. [Ko]), q_1 is the p -th coordinate of the maximal root in the basis of simple roots.⁴

Let $B = e_p \Pi e_p$. According to [MOV], [Me] the algebra B is generated over $\mathbb{C}[x_1, \dots, x_r]$ by U_k , $k = 1, \dots, m$, with defining relations

$$U_k(U_k - x_{i_1(k)}) \dots (U_k - x_{i_1(k)} - \dots - x_{i_{d_k-1}(k)}) = 0,$$

where $i_1(k), \dots, i_{d_k-1}(k)$ are the vertices of the k -th leg of Q enumerated from the nodal vertex, and

$$\sum_{k=1}^m U_k = -x_p.$$

⁴Another context in which the numbers q_1, q_2 appear is the following: the Hilbert series of the invariants $\mathbb{C}[x, y]^\Gamma$ in $\mathbb{C}[x, y]$ under the double cover $\Gamma \subset SL(2)$ of G is

$$\frac{1+t^h}{(1-t^{2q_1})(1-t^{2q_2})}.$$

Namely, the elements U_k are just the elements $a_k^* a_k$, where a_k are the edges of \overline{Q} starting at p and going along the k -th leg.

This implies that the algebra $B_\lambda^\mu := e_p \Pi_\lambda^\mu e_p$ is generated over $\mathbb{C}[z]$ by U_k with the same defining relations, in which $x_i = \lambda_i + \mu_i z$.⁵

Let μ be a weight such that $(\alpha, \mu) \neq 0$ for any positive root α that involves x_p with a strictly positive coefficient. Also, for a number n , let $[n]_q := \frac{1-q^n}{1-q}$. For brevity, if $q = t^2$, we will simply write $[n]$ instead of $[n]_q$.

Theorem 3.3. (i) The algebra $B[z]$ is a finitely generated free module over the algebra K_μ .

(ii) The associated graded algebra of B_λ^μ under the natural filtration is B_0^μ .

(iii) The algebra B_0^μ is Gorenstein. We have $z^{h-1} = 0$, and for generic μ the socle of B_0^μ is spanned by z^{h-2} .

(iv) The Hilbert polynomial of B_0^μ is $E(t) = [h/2][q_1][q_2]$.

(v) The dimension of B_λ^μ is $h q_1 q_2 / 2$, and for generic λ it is semisimple.

Theorem 3.3 is proved in Section 5.

Remark. If μ is regular, then this theorem easily follows from Theorem 3.2, but our statement is more general, so it requires a separate proof.

Considering the case $\mu = e_p$, we immediately get the following corollary. For any numbers λ_{ik} , $i = 1, \dots, d_k$, $k = 1, \dots, m$, define the algebra $B(\lambda)$ generated by U_k and a central element z with defining relations

$$\prod_{i=1}^{d_k} (U_k - \lambda_i) = 0,$$

$$\sum_{k=1}^m U_k = z.$$

Corollary 3.4. (i) $\text{gr} B(\lambda) = B(0)$.

(ii) $B(0)$ is a Gorenstein algebra, and $z^{h-1} = 0$ in $B(0)$.

(iii) The Hilbert polynomial of $B(0)$ is $[h/2][q_1][q_2]$.

Indeed, $B(\lambda)$ is obtained from $B_{\lambda'}^{e_p}$ by a straightforward change of variables.

For comparison note that the Hilbert polynomial of the algebra $B_0 := B(0)/(z)$, according to [MOV], equals $[q_1][q_2]$.

Remark. We have checked using the Magma computer algebra system that the socle of $B(0)$ is spanned by z^{h-2} .

4. THE QUANTUM HEISENBERG ALGEBRA

4.1. Definition and properties of the quantum Heisenberg algebra. The proofs of the main results of this paper are based on the idea of [MOV]:

⁵Note that in our normalization, the degree of all generators of this algebra, including z , is 2.

algebras related to quivers may be obtained from algebras in the category of representations of quantum $SL(2)$ by application of tensor functors into R -bimodules.

We are going to use the notions and notation from [BaKi, EO]. Namely, for every $q \in \mathbb{C}^*$, denote by $\tilde{\mathcal{C}}_q$ the tensor category of finite dimensional comodules over the quantum function algebra $F_q(SL(2))$. If q is not a root of unity of order > 2 , then this category is semisimple and has simple objects V_i , $i \in \mathbb{Z}_+$ (representations with highest weight i) such that V_0 is the neutral object, with the Clebsch-Gordan tensor product rule

$$V_i \otimes V_j = \bigoplus_{n=0}^{\min(i,j)} V_{2n+|i-j|}.$$

On the other hand, if q is a root of unity of order $n > 2$, then the category $\tilde{\mathcal{C}}_q$ is not semisimple. However, the objects V_i are still well defined (the so called dual Weyl modules, i.e. the homogeneous components of the quantum symmetric algebra), and they are simple if $i \leq \bar{n} - 1$, where $\bar{n} = n$ if n is odd and $\bar{n} = n/2$ if n is even.

Consider the tensor algebra $T(V_1 \oplus V_0)$ in $\tilde{\mathcal{C}}_q$, an ind-object in $\tilde{\mathcal{C}}_q$. We will regard it as a graded algebra in which V_1 has degree 1 and V_0 has degree 2. Then we have two morphisms from V_0 to $T(V_1 \oplus V_0)[2]$: the map f which is a composition $V_0 \rightarrow V_1 \otimes V_1 \rightarrow T(V_1 \oplus V_0)[2]$ ⁶, and the map $g : V_0 \rightarrow T(V_1 \oplus V_0)[2]$ coming from the embedding $V_0 \rightarrow V_1 \oplus V_0$. There are also two obvious embeddings $f_1, f_2 : V_0 \otimes V_1 \rightarrow T(V_0 \oplus V_1)[3]$ corresponding to multiplication in two different orders (more precisely, f_1 is the multiplication map, and f_2 is the multiplication map composed with the canonical isomorphism $V_0 \otimes V_1 \rightarrow V_1 \otimes V_0$). Let $\tilde{\mathcal{J}}$ be the ideal generated by the images of $f - g$ and $f_1 - f_2$.

Definition 4.1. The quantum Heisenberg algebra in $\tilde{\mathcal{C}}_q$ is the algebra $\tilde{A} = T(V_0 \oplus V_1)/\tilde{\mathcal{J}}$.

More explicitly, the algebra \tilde{A} is the usual quantum Heisenberg algebra generated by x, y, z with defining relations saying that the element $z = xy - qyx$ is central⁷, i.e.

$$xy - qyx = z, \quad xz - zx = 0, \quad yz - zy = 0,$$

and the usual coaction of the Hopf algebra $F_q(SL(2))$; it is convenient to express this coaction as an action of $U_q(sl(2))$ given by

$$\begin{aligned} ex = y, \quad ey = 0, \quad fy = x, \quad fx = 0, \quad q^h y = qy, \quad q^h x = q^{-1}x, \\ ez = fz = 0, \quad q^h z = z. \end{aligned}$$

⁶Here the first map is a generator of the space $\text{Hom}(V_0, V_1 \otimes V_1)$, and the second map is induced by multiplication.

⁷Here x, y is a weight basis of V_1 , and z is a basis element of V_0 .

(If q is a root of unity, one should add that the divided powers of e and f act by zero). Here the coproduct of $U_q(sl(2))$ is given by $\Delta(e) = e \otimes q^h + 1 \otimes e$, $\Delta(f) = f \otimes 1 + q^{-h} \otimes f$, $\Delta(q^h) = q^h \otimes q^h$.

It is not hard to show that the elements $y^i x^j z^m$, $i, j, m \geq 0$, form a basis of \tilde{A} , and the multiplication in this basis is derived from the commutation relation

$$x^p y^j = \sum_{i=0}^p q^{(j-i)(p-i)} \binom{p}{i}_q \prod_{s=1}^i [j-s+1]_q \cdot y^{j-i} x^{p-i} z^i,$$

where $\binom{p}{i}_q := [p]_q! / [i]_q! [p-i]_q!$, and $[p]_q! := [1]_q \dots [p]_q$.

Now let us consider the structure of the homogeneous subspaces $\tilde{A}[j]$ as $F_q(SL(2))$ -comodules.

Lemma 4.2. (i) If q is not a root of unity (of order > 1) then, as an $F_q(SL(2))$ -comodule, $\tilde{A}[j]$ is isomorphic to the direct sum $V_j \oplus V_{j-2} \oplus \dots \oplus V_{\bar{j}}$, where \bar{j} is j modulo 2.

(ii) If q is a root of unity and the order of q is $2h$, where $h \geq 2$, then the decomposition of (i) holds for $j < h$. Furthermore, for $j = h + p - 1$ with $h - 1 \geq p \geq 1$, the representation $z^p \tilde{A}[j - 2p] = V_{h-p-1} \oplus V_{h-p-3} \oplus \dots$ is canonically contained in $\tilde{A}[j]$.

(iii) In the situation of (ii) let $j < h$, and $E_{i,s}$ be the canonical copy of V_{i-2s} in $\tilde{A}[i]$. Then the image Y_j of the multiplication map $E_{j,0} \otimes E_{1,0} \rightarrow \tilde{A}[j]$ contains $E_{j+1,1}$.

Proof. Parts (i) and (ii) are straightforward by looking at the characters of the representations.

Let us prove part (iii). It is easy to see that

$$(2) \quad xy^j = q^j y^j x + [j]_q y^{j-1} z.$$

On the other hand,

$$fe(y^j x) = f(y^{j+1}) = \sum_{s=0}^j q^{-s} y^s x y^{j-s}.$$

Therefore, using equation (2), we get

$$fe(y^j x) = \sum_{s=0}^j q^{-s} y^s (q^{j-s} y^{j-s} x + [j-s]_q y^{j-s-1} z),$$

which yields

$$(3) \quad fe(y^j x) = \frac{q^{j+1} - q^{-j-1}}{q - q^{-1}} y^j x + \frac{[j]_q [j+1]_{q^{-1}}}{[2]_{q^{-1}}} y^{j-1} z.$$

Since the second coefficient is not zero, and $y^j x \in Y_j$, we get that $y^{j-1} z \in Y_j$ (as Y_j is a subrepresentation of the quantum group). But $y^{j-1} z$ is the highest weight vector of $E_{j+1,1}$, which yields (iii). \square

Now let $q = e^{\pi i/h}$, where $h \geq 2$ is an integer. Then the category $\tilde{\mathcal{C}}_q$ contains a non-abelian monoidal subcategory \mathcal{T}_q of tilting modules, which is closed under taking direct summands. Furthermore, the category \mathcal{T}_q contains a tensor ideal \mathcal{I} , such that $\mathcal{T}_q/\mathcal{I} = \mathcal{C}_q$ is a semisimple tensor category (the fusion category). The simple objects in \mathcal{C}_q are V_0, \dots, V_{h-2} , and the tensor product is given by the Verlinde rule:

$$V_i \otimes V_j = \bigoplus_{n=0}^{\min(i,j,h-2-\max(i,j))} V_{2n+|i-j|}.$$

Consider now the tensor algebra $T(V_0 \oplus V_1)$ in the fusion category \mathcal{C}_q . Then we can define the ideal \mathcal{J} in this algebra generated by the images of the maps $f - g$ and $f_1 - f_2$, where f_1, f_2, f, g are defined in the same way as above.

Definition 4.3. The quantum Heisenberg algebra in \mathcal{C}_q is the algebra $A = T(V_0 \oplus V_1)/\mathcal{J}$.

Remark. The quotient A_0 of A by the ideal generated by the unique copy of V_0 in degree 2 is the quantum symmetric algebra considered in [MOV]; as an object of \mathcal{C}_q it is $V_0 \oplus V_1 \oplus \dots \oplus V_{h-2}$, and the degree of V_i is i .

Proposition 4.4. (i) The degree n component $A[n]$ of A is the object $\bigoplus_{j \leq s/2} V_{s-2j}$, where $s = \min(n, 2h-4-n)$.

(ii) The algebra A is Gorenstein in the following sense: the highest non-trivial degree in A is $A[2h-4] = V_0$, and the multiplication map $A[i] \otimes A[2h-4-i] \rightarrow A[2h-4]$ defines an isomorphism $A[i] \rightarrow A[2h-4-i]^*$.

Proof. Let L be the ideal in \tilde{A} generated by the copy of $V_{h-1} \subset V_1^{\otimes h-1} \subset T(V_0 \oplus V_1)[h-1]$. It is obvious that for any $j = 0, \dots, h-1$, $L[h-1+j]$ has trivial intersection with the canonical subrepresentation $V_{h-3-j} \oplus V_{h-5-j} \oplus \dots$. On the other hand, it follows from Lemma 4.2(iii) that $L[h-1+j]$ contains the canonical copy of V_{h-1-j} for all $0 \leq j \leq h-1$.

By looking at the associated graded algebra of \tilde{A} under the filtration defined by $\deg x = \deg y = 1$, $\deg z = 0$, this implies that $L[h-1+j]$ is a complement of $V_{h-3-j} \oplus V_{h-5-j} \oplus \dots$ in $\tilde{A}[h-1+j]$.

Therefore, the graded algebra \tilde{A}/L , as an object of the category $\tilde{\mathcal{C}}_q$, has the structure specified in part (i) of the proposition. In particular, \tilde{A}/L belongs to the category of tilting modules \mathcal{T}_q . In the category \mathcal{T}_q , the algebra \tilde{A}/L can be written as $T(V_0 \oplus V_1)/\tilde{L}$, where \tilde{L} is the preimage of L in $T(V_0 \oplus V_1)$. Note that both the algebra $T(V_0 \oplus V_1)$ and the ideal \tilde{L} belong to \mathcal{T}_q .

Now we can consider the image A' of \tilde{A}/L in the fusion category \mathcal{C}_q (which is by definition a quotient category of \mathcal{T}_q). We have $A' = T(V_0 \oplus V_1)/J'$,

where J' is the image of \tilde{L} . Since the image of V_{h-1} in \mathcal{C}_q is zero, we have $J' = J$ and $A' = A$, hence (i).

To prove part (ii), it suffices to note that by Lemma 4.2(iii), if $V_j \subset A[i]$, $i \geq j > 0$ then the map $V_j \otimes V_1 \rightarrow V_{j-1}$ defined by multiplication by the generating copy of V_1 and then projecting to $V_{j-1} \subset A[i+1]$ is nonzero. More precisely, this statement follows from the lemma by multiplying by $z^{(i-j)/2}$. \square

4.2. The quantum Heisenberg algebra and the central extension of the preprojective algebra. Now assume that $q = e^{\pi i/h}$, where h is the Coxeter number of Q . Recall (see [EO, MOV]) that there exists a unique tensor functor $\mathcal{F} : \mathcal{C}_q \rightarrow R\text{-bimod}$ from \mathcal{C}_q to the category of R -bimodules, such that the bimodule $\mathcal{F}(V_1)$ is the edge space E of the doubled quiver \overline{Q} . It is checked in [MOV] that $\mathcal{F}(A_0)$ is the preprojective algebra Π_0 .

Proposition 4.5. $\mathcal{F}(A) = \Pi_0^\rho$.

Proof. The algebra Π_0^ρ is the quotient of the path algebra $\mathbb{C}[z]\overline{Q}$ by the relations saying that $\sum_{a \in Q} [a, a^*] = z$, and z is central. It is easy to show, similarly to [MOV], that these relations are images under \mathcal{F} of the relations $xy - qyx = z$, $xz = zx$, $yz = zy$. The proposition is proved. \square

Corollary 4.6. *The Hilbert series of Π_0^ρ is given by the formula in Theorem 3.2, (iv). In particular, the dimension of Π_0^ρ is $h^2(h+1)r/12$.*

Proof. In the Grothendieck ring of \mathcal{C}_q , we have $V_j = P_j(V_1)$, where P_j is the Tchebysheff polynomial of the second kind: $P_j(2 \cos x) = \frac{\sin(j+1)x}{\sin x}$. This means that by Proposition 4.4, (i), the Grothendieck-group-valued Hilbert polynomial of A is

$$\sum_{j=0}^{h-2} \sum_{i=0}^{h-2-j} t^{2i+j} P_j(V_1),$$

and hence by Proposition 4.5, the Hilbert polynomial of Π_0^ρ is

$$\tilde{H}(t) = \sum_{j=0}^{h-2} \sum_{i=0}^{h-2-j} t^{2i+j} P_j(C) = \sum_{j=0}^{h-2} \frac{t^j - t^{2(h-1)-j}}{1-t^2} P_j(C) = \frac{H_0(t) - t^{2h-2} H_0(t^{-1})}{1-t^2}.$$

Thus by formula (1),

$$\tilde{H}(t) = \frac{1 - t^{2h}}{(1-t^2)(1-Ct+t^2)},$$

as desired. Thus we have $\tilde{H}(t) = \frac{1-Pt^h}{1-t^2} H_0(t)$, and hence $\dim \Pi_0^\rho = h \dim \Pi_0/2 = h^2(h+1)r/12$. The corollary is proved. \square

Remark. Since by [MOV], $\mathcal{F}(A_0) = \Pi_0$, we conclude from Proposition 4.4 that the algebra Π_0 , and hence B_0 , are Frobenius algebras.

5. PROOFS OF THEOREMS 3.1,3.2,3.3

5.1. Proofs of Theorems 3.1,3.2. We now prove Theorems 3.1,3.2.

Lemma 5.1. $\Pi[z]$ is a free module over $\mathbb{C}[x_1 - z, \dots, x_r - z]$.

Proof. The fiber of this module at $x_i - z = \lambda_i$ is Π_λ^ρ . Thus, since $\Pi[z]$ is \mathbb{Z}_+ -graded, it suffices to show that the dimension of Π_λ^ρ is at least as big as that of Π_0^ρ , for generic λ . To do so, note that by Theorem 1.2 in [CB1], for generic λ the algebra Π_λ^ρ has an irreducible representation with dimension vector being each positive root α . Therefore,

$$\dim \Pi_\lambda^\rho \geq \sum_{\alpha > 0} (\alpha, \rho)^2 = h(\rho, \rho) = h^2 \frac{\dim \mathfrak{g}}{12} = \frac{h^2(h+1)r}{12},$$

which together with Corollary 4.6 implies the desired inequality. \square

Lemma 5.1 together with Corollary 4.6 imply part (ii) of Theorem 3.1.

Now, by Proposition 4.4, Proposition 4.5, and Lemma 2.1, Π_0^ρ is a Frobenius algebra. Hence by Proposition 2.4 (i) it is a Gorenstein algebra. Therefore, since $\Pi[z]$ is a free module over $\mathbb{C}[x_1 - z, \dots, x_r - z]$, by Proposition 2.4 (ii), the algebra $\Pi[z]$ is Gorenstein, and hence again by Proposition 2.4 (ii), Π is Gorenstein. This implies part (i) of Theorem 3.1. Thus Theorem 3.1 is proved.

Now let μ be a regular weight. Since Π_0 is finite dimensional, the algebra Π_0^μ is finitely generated as a module over $\mathbb{C}[z]$. Moreover, if $z_0 \in \mathbb{C}$, we have $\Pi_0^\mu / (z - z_0) = \Pi_{z_0\mu}$, which is zero by Theorem 1.2 of [CB1] for any $z_0 \neq 0$. Hence Π_0^μ is finite dimensional. This implies that $\Pi[z]$ is a finitely generated K_μ -module (since this module is \mathbb{Z}_+ -graded and its zero-fiber is Π_0^μ). Since $\Pi[z]$ is a Gorenstein algebra, it is in particular Cohen-Macaulay, and by Proposition 2.3 (i), $\Pi[z]$ is a free K_μ -module, hence parts (i),(ii) of Theorem 3.2.

We also see that the Hilbert polynomial of Π_0^μ is the same as that for Π_0^ρ , so Corollary 4.6 implies Theorem 3.2, (iv), and the dimension formula of (v). The semisimplicity of Π_λ^μ follows from the fact that for generic λ , by Theorem 1.2 of [CB1], it has an irreducible representation with dimension vector being every positive root α ; as we've shown, the sum of dimensions of these representations is $h^2(h+1)r/12 = \dim \Pi_0^\mu$.

We also see, by Proposition 2.4 (ii), that Π_0^μ is Gorenstein (=Frobenius). This proves the first statement of Theorem 3.2, (iii). The second statement follows from part (iv), and the third statement (the socle is $z^{h-2}R$) follows from the fact that this is so for $\mu = \rho$ (by Proposition 4.4 and Proposition 4.5). Theorem 3.2 is proved.

Remark. Let N be the ideal in Π_0^μ generated by z , and $H_k(t)$ be the Hilbert polynomial of the quotient N^k/N^{k+1} . The above arguments show

that for generic μ one has

$$\sum_{k \geq 0} H_k(t) u^k = \frac{H_0(t) - ut^h P H_0(ut)}{1 - ut^2}.$$

Indeed, if $\mu = \rho$ then using Propositions 4.4 and 4.5 as in the proof of Corollary 4.6, we get

$$\begin{aligned} \sum_{k \geq 0} H_k(t) u^k &= \sum_{j=0}^{h-2} \sum_{i=0}^{h-2-j} t^{2i+j} u^j P_j(C) = \\ \sum_{j=0}^{h-2} \frac{t^j - t^{2(h-1)-j} u^{h-1-j}}{1 - ut^2} P_j(C) &= \frac{H_0(t) - t^{2h-2} u^{h-1} H_0(t^{-1} u^{-1})}{1 - ut^2}, \end{aligned}$$

which gives the desired formula. Also, it is clear from this argument that for $\mu = \rho$ the operator $z : \Pi_0^\mu[j] \rightarrow \Pi_0^\mu[j+2]$ has maximal rank for all j . Hence, the same formula applies to generic μ .

5.2. Proof of Theorem 3.3. It follows from Theorems 3.1, 3.2 that the algebras B and $B[z]$ are Gorenstein. Since B_0 is finite dimensional, the algebra B_0^μ is finitely generated as a module over $\mathbb{C}[z]$. Moreover, by Theorem 1.2 of [CB1], z must act by zero in every irreducible representation of B_0^μ . Thus, B_0^μ is finite dimensional. This implies that $B[z]$ is a finitely generated K_μ -module. Since $B[z]$ is a Gorenstein algebra, it is in particular Cohen-Macaulay, and by Proposition 2.3 (i), $B[z]$ is a free K_μ -module, hence parts (i), (ii) of Theorem 3.3.

We also see that the Hilbert polynomial of B_0^μ is the same as that for B_0^ρ , so Proposition 4.6 implies Theorem 3.3, (iv), and the dimension formula of (v). The semisimplicity of B_λ^μ follows from the fact that for generic λ , by Theorem 1.2 of [CB1], it has an irreducible representation corresponding to every positive root α that involves x_p ; the sum of dimensions of these representations is $hq_1 q_2 / 2 = \dim B_0^\mu$.

We also see, by Proposition 2.4 (ii), that B_0^μ is Gorenstein (=Frobenius). This proves the first statement of Theorem 3.3, (iii). The second statement follows from part (iv), and the third statement (the socle is z^{h-2}) follows from the fact that this is so for $\mu = \rho$ (by Proposition 4.4 and Proposition 4.5). Theorem 3.3 is proved.

6. RELATION TO HECKE ALGEBRAS OF 2-DIMENSIONAL COMPLEX REFLECTION GROUPS

Let us discuss the connection of the above results with the theory of Hecke algebras for complex reflection groups, due to Broué, Malle, and Rouquier ([BMR]).

6.1. Finiteness of cyclotomic Hecke algebras in 2 dimensions. Let Γ be a 2-dimensional irreducible complex reflection group, and H be its cyclotomic Hecke algebra (see [BMR]); it is defined by generators and relations which are deformations of the relations of Γ . The algebra H is a module over the algebra $\mathbb{C}[\mathbf{T}]$ of functions on the torus \mathbf{T} of parameters. Denote by $H(\Lambda)$ the specialization of H at a point $\Lambda \in \mathbf{T}$.

It is known ([BMR]) that the dimension of $H(\Lambda)$ is generically $\geq |\Gamma|$. Moreover, it is conjectured (see [BMR]), that the algebra H is a free $\mathbb{C}[\mathbf{T}]$ -module, and hence $H(\Lambda)$ has dimension exactly $|\Gamma|$ for all values of Λ , i.e. it is a flat deformation of the group algebra $\mathbb{C}[\Gamma]$. As far as we know, this conjecture has been checked by J. Müller using a computer for all cases except G_{17}, G_{18}, G_{19} (see [GGOR], Remark 5.12).

Here we give a proof of a weak version of this conjecture.

Theorem 6.1. *The algebra H is a finitely generated module over $\mathbb{C}[\mathbf{T}]$. Therefore, the algebra $H(\Lambda)$ is finite dimensional for all Λ , and has dimension $|\Gamma|$ for generic Λ , and $\geq |\Gamma|$ for all Λ .*

The rest of the subsection is devoted to the proof of this theorem⁸.

The theorem is known for the infinite series of complex reflection groups ([BMR]), so it is sufficient to concentrate on the exceptional ones, i.e. the groups G_n in the Shephard-Todd classification [ST], $4 \leq n \leq 22$.

For the proof we need explicit presentations of the groups Γ . Recall that in all cases the image G of Γ in $PGL_2(\mathbb{C})$ is the tetrahedral group, octahedral group, or icosahedral group. Thus the groups G_4, \dots, G_{22} fall into three families: tetrahedral, octahedral, and icosahedral. In each family, there is a maximal group \hat{G} of order $|G|^2$, which is a central extension of G by a cyclic group of order $|G|$ (generated by an element Z), and all others are its subgroups.

The realizations of the particular groups G_n , $4 \leq n \leq 22$, are as follows.

The tetrahedral family.

The tetrahedral group G : $a^2 = b^3 = c^3 = 1$, $abc = 1$ (order 12).

The maximal group $\hat{G} = G_7$ (order 144): $a_*^2 = b_*^3 = c_*^3 = 1$, $a_*b_*c_* = Z$, Z is central; conjugacy classes of reflections are powers of a_*, b_*, c_* .

Group G_4 (order 24) is generated by $a = a_*Z^{-3}, b = b_*Z^2, c = c_*$. The relations are $a^2 = \zeta^{-1}, b^3 = \zeta, c^3 = 1, abc = 1$, ζ is central (here $\zeta = Z^6$). Conjugacy classes of reflections are powers of c .

Group G_5 (order 72) is generated by $a = a_*Z^{-1}, b = b_*, c = c_*$. The relations are $a^2 = \zeta^{-1}, b^3 = 1, c^3 = 1, abc = 1$, ζ is central (here $\zeta = Z^2$). Conjugacy classes of reflections are powers of b, c .

Group G_6 (order 48) is generated by $a = a_*, b = b_*Z^{-1}, c = c_*$. The relations are $a^2 = 1, b^3 = \zeta^{-1}, c^3 = 1, abc = 1$, ζ is central (here $\zeta = Z^3$). Conjugacy classes of reflections are powers of a, c .

Octahedral family.

⁸The first author is very grateful to R. Rouquier for help with this proof.

The octahedral group G : $a^2 = b^3 = c^4 = 1$, $abc = 1$ (order 24).

The maximal group $\widehat{G} = G_{11}$ (order 576): $a_*^2 = b_*^3 = c_*^4 = 1$, $a_*b_*c_* = Z$, Z is central; conjugacy classes of reflections are powers of a_* , b_* , c_* .

Group G_8 (order 96) is generated by $a = a_*Z^{-3}$, $b = b_*Z^2$, $c = c_*$. The relations are $a^2 = \zeta^{-1}$, $b^3 = \zeta$, $c^4 = 1$, $abc = 1$, ζ is central (here $\zeta = Z^6$). Conjugacy classes of reflections are powers of c .

Group G_9 (order 192) is generated by $a = a_*$, $b = b_*Z^{-1}$, $c = c_*$. The relations are $a^2 = 1$, $b^3 = \zeta^{-1}$, $c^4 = 1$, $abc = 1$, ζ is central (here $\zeta = Z^3$). Conjugacy classes of reflections are powers of a , c .

Group G_{10} (order 288) is generated by $a = a_*Z^{-1}$, $b = b_*$, $c = c_*$. The relations are $a^2 = \zeta^{-1}$, $b^3 = 1$, $c^4 = 1$, $abc = 1$, ζ is central (here $\zeta = Z^2$). Conjugacy classes of reflections are powers of b , c .

Group G_{12} (order 48) is generated by $a = a_*$, $b = b_*Z^{-4}$, $c = c_*Z^3$. The relations are $a^2 = 1$, $b^3 = \zeta^{-1}$, $c^4 = \zeta$, $abc = 1$, ζ is central (here $\zeta = Z^{12}$). Conjugacy classes of reflections are powers of a .

Group G_{13} (order 96) is generated by $a = a_*$, $b = b_*Z^2$, $c = c_*Z^{-3}$, and $f = c_*^2$. The relations are $a^2 = 1$, $b^3 = \zeta$, $f^2 = 1$, $c^2\zeta = f$, $abc = 1$, ζ is central (here $\zeta = Z^6$). Conjugacy classes of reflections are powers of a and f .

Group G_{14} (order 144) is generated by $a = a_*$, $b = b_*$, $c = c_*Z^{-1}$. The relations are $a^2 = 1$, $b^3 = 1$, $c^4 = \zeta^{-1}$, $abc = 1$, ζ is central (here $\zeta = Z^4$). Conjugacy classes of reflections are powers of a , b .

Group G_{15} (order 288) is generated by $a = a_*$, $b = b_*$, $c = c_*Z^{-1}$, $f = c_*^2$. The relations are $a^2 = 1$, $b^3 = 1$, $f^2 = 1$, $c^2\zeta = f$, $abc = 1$, ζ is central (here $\zeta = Z^2$). Conjugacy classes of reflections are powers of a , b , and f .

Icosahedral family.

The icosahedral group G : $a^2 = b^3 = c^5 = 1$, $abc = 1$ (order 60).

The maximal group $\widehat{G} = G_{19}$ (order 3600): $a_*^2 = b_*^3 = c_*^5 = 1$, $a_*b_*c_* = Z$, Z is central; conjugacy classes of reflections are powers of a_* , b_* , c_* .

Group G_{16} (order 600) is generated by $a = a_*Z^{-3}$, $b = b_*Z^2$, $c = c_*$. The relations are $a^2 = \zeta^{-1}$, $b^3 = \zeta$, $c^5 = 1$, $abc = 1$, ζ is central (here $\zeta = Z^6$). Conjugacy classes of reflections are powers of c .

Group G_{17} (order 1200) is generated by $a = a_*$, $b = b_*Z^{-1}$, $c = c_*$. The relations are $a^2 = 1$, $b^3 = \zeta^{-1}$, $c^5 = 1$, $abc = 1$, ζ is central (here $\zeta = Z^3$). Conjugacy classes of reflections are powers of a , c .

Group G_{18} (order 1800) is generated by $a = a_*Z^{-1}$, $b = b_*$, $c = c_*$. The relations are $a^2 = \zeta^{-1}$, $b^3 = 1$, $c^5 = 1$, $abc = 1$, ζ is central (here $\zeta = Z^2$). Conjugacy classes of reflections are powers of b , c .

Group G_{20} (order 360) is generated by $a = a_*Z^{-5}$, $b = b_*$, $c = c_*Z^4$. The relations are $a^2 = \zeta^{-1}$, $b^3 = 1$, $c^5 = \zeta^2$, $abc = 1$, ζ is central (here $\zeta = Z^{10}$). Conjugacy classes of reflections are powers of b .

Group G_{21} (order 720) is generated by $a = a_*$, $b = b_*$, $c = c_*Z^{-1}$. The relations are $a^2 = 1$, $b^3 = 1$, $c^5 = \zeta^{-1}$, $abc = 1$, ζ is central (here $\zeta = Z^5$). Conjugacy classes of reflections are powers of a , b .

Group G_{22} (order 240) is generated by $a = a_*, b = b_* Z^5, c = c_* Z^{-6}$. The relations are $a^2 = 1, b^3 = \zeta, c^5 = \zeta^{-2}, abc = 1, \zeta$ is central (here $\zeta = Z^{15}$). Conjugacy classes of reflections are powers of a .

Note that in each case, conjugacy classes of reflections are represented by those of the generators g for which one of the defining relations is $g^p = 1$. The braid groups B_Γ of Γ are obtained by removing such relations, and thus the Hecke algebras are obtained by replacing them with the relations $(g - \mathbf{b}_1) \dots (g - \mathbf{b}_p) = 0$, where $\mathbf{b}_1, \dots, \mathbf{b}_p$ are invertible parameters (i.e., characters of \mathbf{T}); see [BMR, Ma].

Now we proceed to prove the theorem. Since it is known that generically $\dim H(\Lambda) \geq |\Gamma|$, it suffices to show that H is a finitely generated module over $\mathbb{C}[\mathbf{T}]$.

To do so, we note that the group G is the group of even elements in a finite Coxeter group of rank 3: of type $A_1 \times I_m$ in the dihedral case (which we are not considering) and A_3, B_3, H_3 for the tetrahedral, octahedral, and icosahedral cases, respectively. Therefore, by Theorem 2.3 of [ER], H is a finite module over $\mathbb{C}[\mathbf{T}][Z, Z^{-1}]$ generated by $|G|$ generators (denoted in [ER] by $T_w, w \in G$). Let S be the support of this module (a subvariety in $\mathbf{T} \times \mathbb{C}^*$). If $(\Lambda, z) \in S$, then $H(\Lambda)$ must have an irreducible representation of dimension $d \leq |G|^{1/2}$, in which Z acts by the scalar z . Taking the determinants of the defining relations of H in this representation, we obtain that z^d is a certain character of \mathbf{T} evaluated at Λ (there are finitely many possibilities for such characters). Thus S is contained in a finite union of subtori of $\mathbf{T} \times \mathbb{C}^*$ whose projections to \mathbf{T} are finite. We conclude that the action of $\mathbb{C}[\mathbf{T}][Z, Z^{-1}]$ in H factors through the algebra $\mathcal{R} := \mathbb{C}[\mathbf{T}][Z, Z^{-1}]/J$ for a certain ideal J whose zero set is S , and the algebra \mathcal{R} is a finitely generated module over $\mathbb{C}[\mathbf{T}]$. This implies the theorem.

6.2. The connection between $B(\lambda)$ and $H(\Lambda)$. Now assume that $\Gamma = \widehat{G}$ is a maximal group, i.e. G_7, G_{11}, G_{19} . In this case, using the notation of Subsection 3.3, the cyclotomic Hecke algebra is generated by elements Y_k , and invertible central variables $\mathbf{b}_{jk}, j = 1, \dots, d_k, k = 1, \dots, m$, with defining relations

$$\prod_{j=1}^{d_k} (Y_k - \mathbf{b}_{jk}) = 0, k = 1, \dots, m,$$

and $\prod_{k=1}^m Y_k = Z$ is central. The specialization $H(\Lambda)$ of the algebra H is obtained when the variables \mathbf{b}_{jk} map to complex numbers Λ_{jk} .

Let $\Lambda = e^{2\pi i \lambda}$. In this case we can define an algebra homomorphism $\phi : H(\Lambda) \rightarrow B(\lambda)$ (the Riemann-Hilbert homomorphism) in the following manner. Let $\zeta_k, k = 0, 1, \dots, m$ be distinct real numbers listed in increasing order. Consider the differential equation

$$\frac{dF}{d\zeta} = \sum_{k=1}^m \frac{U_k F}{\zeta - \zeta_k}$$

with respect to a holomorphic function $F(\zeta)$ with values in $B(\lambda)$, and define $\phi(Y_k)$ to be the monodromy operator of this equation around a loop γ_k which starts and ends at ζ_0 and goes around ζ_k counterclockwise, passing $\zeta_1, \dots, \zeta_{k-1}$ from below. It is easy to check that this gives rise to a well defined homomorphism ϕ .

For any $s \in \mathbb{C}$, let $H(\Lambda)_s$ denote the generalized eigenspace of Z with eigenvalue $e^{2\pi is}$. It is a direct summand subalgebra of $H(\Lambda)$. Similarly, denote by $B(\lambda)_s$ the direct summand subalgebra in $B(\lambda)$ which is the generalized eigenspace of z with eigenvalue s . It is clear that for any s , ϕ maps $H(\Lambda)_s$ to $B(\lambda)_s$.

Also, let $H_* := H(1)/(Z - 1)$, and $B_0 = B(0)/(z) = e_p \Pi_0 e_p$. Since $\phi(Z) = e^{2\pi iz}$, the map ϕ descends to a homomorphism $\phi_* : H_* \rightarrow B_0$.

Proposition 6.2. (i) ϕ_* is an isomorphism.

(ii) $\phi_0 : H(1)_0 \rightarrow B(0)$ is an isomorphism. In particular, the block of the trivial representation of $H(1)$ (i.e. the 1-dimensional representation where all the generators act by 1) is equivalent to the category of representations of $B(0)$.

(iii) There exists $\varepsilon > 0$ such that if $|\lambda_{jk}| < \varepsilon$ for all j, k and $|s| < \varepsilon$, then ϕ defines an isomorphism $H(\Lambda)_s \rightarrow B(\lambda)_s$.

Proof. It is clear that $\phi(Y_k - 1) = 2\pi i U_k + h.d.t.$, where $h.d.t.$ denotes the higher degree terms in U_k . Thus the homomorphism $\phi : H(1) \rightarrow B(0)$ is surjective, and hence ϕ_* is surjective. On the other hand, it is clear that $\phi(Z) = e^{2\pi iz}$, so ϕ factors through $H(1)_0$ where it descends to ϕ_0 . So ϕ_0 is surjective.

Now let us show that ϕ_* is an isomorphism. To do so, recall that the dimension of B_0 is $|G|$ (see e.g. [MOV]). Thus it suffices to show that $\dim H_* \leq |G|$. But this is trivial for type A and follows from [ER], Theorem 2.3, applied to finite Coxeter groups of rank 3, for types D and E . So (i) is proved.

To prove (ii), it remains to show that ϕ_0 is an isomorphism. For this purpose note that by Schur's lemma, any irreducible representation of $H(1)_0$ factors through H_* and hence by (i) is 1-dimensional, with all generators acting by 1. This means that $H(1)_0 = \lim_{n \rightarrow \infty} \text{proj}_{J^n} H(1)$, where J is the ideal generated by the elements $Y_k - 1$. So we can define elements $U'_k = (2\pi i)^{-1} \log(Y_k) \in H(1)_0$ (using the power series for the logarithm). The elements U'_k satisfy the equation $(U'_k)^{d_k} = 0$, and $\sum_k U'_k + h.d.t. = 0$, where $h.d.t.$ stand for higher degree expression in U'_k . Since U'_k obviously generate $H(1)_0$, we can put on $H(1)_0$ a decreasing filtration defined by $\deg(U'_k) = 1$. Then we find that $\text{gr} H(1)_0$ is a quotient of $B(0)$, which implies that $\dim H(1)_0 \leq \dim B(0)$. This implies (ii).

Finally, for proof of (iii) it suffices to note that

- 1) $B(\lambda)$ is a flat deformation of $B(0)$, so $\dim B(\lambda) = \dim B(0)$;
- 2) There exists a constant $K > 0$ such that for any $\varepsilon > 0$, if $|\lambda_{jk}| < \varepsilon$ for all j, k , then $B(\lambda) = \bigoplus_{s: |s| < K\varepsilon} B(\lambda)_s$,

- 3) The map $\phi_s : H(\Lambda)_s \rightarrow B(\lambda)_s$ is surjective; and
 4) by Theorem 6.1, there is $\varepsilon > 0$ such that if $|\lambda_{jk}| < \varepsilon$ for all j, k , then the dimension of the direct sum of $H(\Lambda)_s$ over s such that $|s| < K\varepsilon$ is $\leq \dim H(1)_0 = \dim B(0)$. \square

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